

Scalar perturbations during multiple field slow-roll inflation

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Abstract

We calculate the scalar gravitational and matter perturbations in the context of slow-roll inflation with multiple scalar fields, that take values on a (curved) manifold, to first order in slow roll. For that purpose a basis for these perturbations determined by the background dynamics is introduced and multiple field slow-roll functions are defined. To obtain analytic solutions to first order, the scalar perturbation modes have to be treated in three different regimes. Consistency of the various approximations fixes their matching times. Multiple field effects in the gravitational potential are due to the rotation of the basis and to the particular solution caused by the coupling to the field perturbation perpendicular to the field velocity. They can contribute even to leading order if the corresponding multiple field slow-roll function is sizable during the last 60 e-folds. The analytical results are illustrated and checked numerically with the example of a quadratic potential.

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1 Introduction

As has been known for a long time, inflation [4, 10] offers a mechanism for the production of density perturbations, which are supposed to be the seeds for the formation of large scale structures in the universe. This mechanism is the magnification of microscopic quantum fluctuations in the scalar fields present during the inflationary epoch into macroscopic matter and metric perturbations. Also, since a part of the primordial spectrum of density perturbations is observed in the cosmic microwave background radiation (CMBR), this mechanism offers one of the most important ways of checking and constraining possible models of inflation, see e.g. [5].

The theory of the production of density perturbations in the case of a single real scalar field has been studied for a long time [2, 18, 8, 30, 16, 29, 28]. However, to realize inflation that leads to the observed density perturbations in a model without very unnatural values of the parameters, it is now thought that one needs more than one field. This is a strong motivation for hybrid inflation models [11] (related models can be found in [12]). Also, many theories beyond the standard model of particle physics, like grand unification, supersymmetry or effective supergravity from string theory, contain a lot of scalar fields. Ultimately one would hope to be able to identify those fields that can act as inflatons. For all these reasons it is important to develop a theory for perturbations from multiple field inflation as well. Work in this direction has been done by several people. Using gauge invariant variables the authors of [23, 22, 3] treated two field inflation. The fluid flow approach was extended to multiple fields in [12], while a more geometrical approach was used in [26, 21]; both methods assumed several slow-roll-like conditions on the potential. Using slow-roll approximations for both the background and the perturbation equations the authors of [19, 24, 6] were able to find expressions for the metric perturbations in multiple field inflation.

In this paper we compute the scalar gravitational and matter perturbations during multiple field inflation to first order in slow roll. We generalize the slow-roll parameters for a single background field to multiple scalar fields in a systematic way without assuming implicitly that slow roll is valid. We obtain a slow-roll formalism that is independent of the specific choice of time variable and valid for the general multiple scalar field case, where the fields may parameterize a (geometrically non-trivial) manifold. We can then give a clear quantification of the relative importance of terms in the equations obtained by extending the single field density perturbation calculations by Mukhanov, Feldman and Brandenberger [18] to multiple fields. During inflation there is a relatively sharp transition in the behaviour of a fluctuation when the corresponding wavelength ‘passes through the horizon’; this moment identifies a certain scale k . For observationally interesting scales (those that reentered the horizon only after the time of recombination when the CMBR was formed) this happened approximately 60 e-folds before the end of inflation [8]. With a careful analysis of this transition region and the subsequent region in the context of the above slow-roll formalism, we derive an explicit expression for the density perturbations to first order in slow roll. In particular we find the explicit multiple field contribution terms. In previous literature the existence and possible importance of some of these terms has not been realized.

Apart from this introduction the paper is structured as follows. In section 2.1 the background with multiple scalar fields is described using geometrical concepts which are explained in appendix A. An orthonormal basis induced by the dynamics of the background fields is also introduced here. Section 2.2 then describes the multiple field slow-roll formalism.

Section 3 is devoted to the perturbations in multiple field inflation and is the main part of this paper. In section 3.1 the equations of motion for the scalar gravitational and matter perturbations are derived, and the choice of perturbation variables is discussed.

The next section 3.2 focuses on the quantization of the dynamical scalar perturbations. After discussing the outlines of the calculation in section 3.3, solving the equations and computing the correlator of the gravitational potential is done in section 3.4. For this calculation the inflationary epoch is split into three regions, which are treated separately. The results for the gravitational potential and its correlator are explicitly expressed in terms of background quantities only, except for the particular solution contribution. How this term can be written in terms of background quantities as well by assuming slow roll for the perturbations is discussed in section 3.5.

In section 4 the example of a quadratic potential with multiple scalar fields is discussed, not only to illustrate the theory of section 3, but also as a numerical check of our analytical results. Analytical expressions for this example are derived in section 4.1, while section 4.2 gives numerical results. The results of this paper are summarized and discussed in section 5.

2 Slow-roll background in multiple field inflation

2.1 Equations of motion for the background

The background of the universe is described by the flat Robertson-Walker metric in terms of a general time variable τ :

$$ds^2 = -b^2 d\tau^2 + a^2 d\mathbf{x}^2 \quad (1)$$

with $a(\tau)$ the spatial scale factor. The temporal scale factor b is defined by the specific choice of time variable: for comoving time t and conformal time η it is given by $b = 1$ and $b = a$, respectively, leading to the relation $dt = ad\eta$. Since different equations are best solved using different time variables, it is convenient to set up the formalism for a general time variable. Moreover, this approach allows us to point out the coordinate independent properties of the formalism, most importantly of the slow-roll approximation. A derivative with respect to the general time variable τ is denoted by $\dot{} \equiv \partial_\tau$, one with respect to comoving time by $\dot{} \equiv \partial_t$, and one with respect to conformal time by $\prime \equiv \partial_\eta$. Hubble parameters $H_a \equiv \partial_\tau a/a$ and $H_b \equiv \partial_\tau b/b$ are associated with the scale factors a and b . For H_a in terms of comoving and conformal time we define the conventional symbols: $H = \dot{a}/a$ and $\mathcal{H} = a'/a = aH$.

For the matter part of the universe we consider scalar fields ϕ that are the coordinates on a possibly non-trivial field manifold \mathcal{M} with metric \mathbf{G} . The Lagrangean for the scalar field theory with a potential V on this manifold in a general spacetime that is quadratic in the derivatives can be written as

$$\mathcal{L}_{\mathcal{M}} = \sqrt{-g} \left(-\frac{1}{2} \partial^\mu \phi \cdot \partial_\mu \phi - V(\phi) \right) = \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi^T \mathbf{G} \partial_\nu \phi - V(\phi) \right), \quad (2)$$

with g the determinant of $g_{\mu\nu}$. Notice that the kinetic term contains both the inverse spacetime metric $g^{\mu\nu}$ and the field metric \mathbf{G} . Definitions of various geometrical concepts like the inner product $\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^\dagger \mathbf{B} = \mathbf{A}^T \mathbf{G} \mathbf{B}$ and the derivatives \mathcal{D}_μ (with respect to spacetime) and ∇ (with respect to the fields), that are covariant with respect to the geometry of the manifold \mathcal{M} , can be found in appendix A.

The equations of motion for the scalars are given by

$$g^{\mu\nu} \left(\mathcal{D}_\mu \delta_\nu^\lambda - \Gamma_{\mu\nu}^\lambda \right) \partial_\lambda \phi - \mathbf{G}^{-1} \nabla^T V = 0, \quad (3)$$

and the Einstein equations read

$$\frac{1}{\kappa^2} G^\mu_\nu = T^\mu_\nu = \partial^\mu \phi \cdot \partial_\nu \phi - \delta^\mu_\nu \left(\frac{1}{2} \partial^\lambda \phi \cdot \partial_\lambda \phi + V \right), \quad (4)$$

with G^μ_ν the Einstein tensor and $\kappa^2 \equiv 8\pi G = 8\pi/M_P^2$. From these formulae (3) and (4) we obtain the background equation of motion for the scalar fields ϕ ,

$$\mathcal{D}\phi^i + 3H_a\phi^i + b^2 \mathbf{G}^{-1} \nabla^T V = 0, \quad (5)$$

and the Friedmann equations

$$H_a^2 = \frac{1}{3}\kappa^2 \left(\frac{1}{2}|\dot{\phi}|^2 + b^2 V \right), \quad \mathcal{D}H_a = -\frac{1}{2}\kappa^2 |\dot{\phi}|^2. \quad (6)$$

Here we have introduced the “slow-roll” derivative \mathcal{D} which is defined as follows: on any quantity A that does not have any b dependence, $\mathcal{D}(b^n A) = (\mathcal{D}_\tau - nH_b)(b^n A)$. In particular this means that $\mathcal{D}\phi^i = (\mathcal{D}_\tau - H_b)\phi^i$, $\mathcal{D}^2\phi^i = (\mathcal{D}_\tau - 2H_b)(\mathcal{D}_\tau - H_b)\phi^i$, $\mathcal{D}H_a = (\partial_\tau - H_b)H_a$, etc. Notice that the slow-roll derivative equals the comoving time derivative \mathcal{D}_t if comoving time is used ($b = 1$), while with conformal time it reads $\mathcal{D} = \mathcal{D}_\eta - n\mathcal{H}$. There are two reasons to introduce this special derivative. In the first place it allows us to define quantities like velocities in a way that does not depend on a specific choice of time variable. In the second place it turns out that applying this derivative to quantities like fields leads to terms that are one order higher in the slow-roll approximation, as we explain in the next section.

We finish this section by introducing a preferred basis $\{\mathbf{e}_n\}$ on the field manifold that is induced by the dynamics of the system. The first unit vector \mathbf{e}_1 is given by the direction of the field velocity $\dot{\phi}^i$. The second unit vector \mathbf{e}_2 points in the direction of that part of the field acceleration $\mathcal{D}\dot{\phi}^i$ that is perpendicular to the first unit vector \mathbf{e}_1 . This Gram-Schmidt orthogonalization process can be extended to any n : the unit vector \mathbf{e}_n points in the direction of $\phi^{(n)} \equiv \mathcal{D}^{(n-1)}\dot{\phi}^i$ that is perpendicular to the first $n-1$ unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$. Using the projection operators \mathbf{P}_n , which project on the \mathbf{e}_n , and \mathbf{P}_n^\perp , which project on the subspace that is perpendicular to $\mathbf{e}_1, \dots, \mathbf{e}_n$, the definitions of the unit vectors are given by

$$\mathbf{e}_n = \frac{\mathbf{P}_{n-1}^\perp \phi^{(n)}}{|\mathbf{P}_{n-1}^\perp \phi^{(n)}|}, \quad \mathbf{P}_n = \mathbf{e}_n \mathbf{e}_n^\dagger, \quad \mathbf{P}_n^\perp = \mathbb{1} - \sum_{q=1}^n \mathbf{P}_q, \quad (7)$$

for all $n = 1, 2, \dots$ and with the definition $\mathbf{P}_0^\perp \equiv \mathbb{1}$. Notice that the unit vectors \mathbf{e}_n will in general depend on time. However, because the slow-roll derivative \mathcal{D} was used in the definition of this basis, the definition does not depend on a specific choice of time variable. By construction the vector $\phi^{(n)}$ can be expanded in these unit vectors as

$$\phi^{(n)} = (\mathbf{P}_1 + \dots + \mathbf{P}_n) \phi^{(n)} = \sum_{p=1}^n \phi_p^{(n)} \mathbf{e}_p, \quad \phi_p^{(n)} = \mathbf{e}_p \cdot \phi^{(n)}. \quad (8)$$

In particular, we have that $\phi_n^{(n)} = \mathbf{e}_n \cdot \phi^{(n)} = |\mathbf{P}_{n-1}^\perp \phi^{(n)}|$. As the projection operators \mathbf{P}_1 and \mathbf{P}_1^\perp turn out to be the most important in our discussions, we introduce the short-hand notation $\mathbf{P}^\parallel = \mathbf{P}_1$ and $\mathbf{P}^\perp = \mathbf{P}_1^\perp = \mathbb{1} - \mathbf{P}^\parallel$. In terms of these two operators we can write a general vector and matrix as $\mathbf{A} = \mathbf{A}^\parallel + \mathbf{A}^\perp$ and $\mathbf{M} = \mathbf{M}^{\parallel\parallel} + \mathbf{M}^{\parallel\perp} + \mathbf{M}^{\perp\parallel} + \mathbf{M}^{\perp\perp}$, with $\mathbf{A}^\parallel \equiv \mathbf{P}^\parallel \mathbf{A}$ and $\mathbf{M}^{\parallel\parallel} \equiv \mathbf{P}^\parallel \mathbf{M} \mathbf{P}^\parallel$, etc.

2.2 Slow roll

Slow-roll inflation is driven by a scalar field potential that is almost flat and therefore acts as an effective cosmological constant. In the case of a single scalar field, the notion of slow roll is well-established (see e.g. [8, 12, 9]). This concept can be generalized to multiple scalar fields in a geometrical way using the unit vectors introduced in the previous section. The

system consisting of (5) and (6) is said to be in the slow-roll regime if the comoving time derivatives satisfy $|\mathcal{D}_t \dot{\phi}| \ll |3H\dot{\phi}|$ and $\frac{1}{2}|\dot{\phi}|^2 \ll V$. A more precise definition not depending on the use of comoving time is given below (13).

We introduce the following functions for an arbitrary time variable τ :

$$\tilde{\epsilon}(\phi) \equiv -\frac{\mathcal{D}H_a}{H_a^2}, \quad \tilde{\eta}^{(n)}(\phi) \equiv \frac{\mathcal{D}^{n-1}\phi^i}{(H_a)^{n-1}|\dot{\phi}|}. \quad (9)$$

We often use the short-hand notation $\tilde{\eta} = \tilde{\eta}^{(2)}$ and $\tilde{\xi} = \tilde{\eta}^{(3)}$. Both these vectors can be decomposed in components parallel ($\tilde{\eta}^{\parallel}, \tilde{\xi}^{\parallel}$) and perpendicular ($\tilde{\eta}^{\perp}$) to the field velocity $\dot{\phi}^i$:

$$\tilde{\eta}^{\parallel} = \mathbf{e}_1 \cdot \tilde{\eta} = \frac{\mathcal{D}\phi^i \cdot \dot{\phi}^i}{H_a|\dot{\phi}|^2}, \quad \tilde{\eta}^{\perp} = \mathbf{e}_2 \cdot \tilde{\eta} = \frac{|\mathcal{D}\phi^i|^{\perp}}{H_a|\dot{\phi}|}, \quad \tilde{\xi}^{\parallel} = \mathbf{e}_1 \cdot \tilde{\xi} = \frac{\mathcal{D}^2\phi^i \cdot \dot{\phi}^i}{H_a^2|\dot{\phi}|^2}. \quad (10)$$

(Even though $\tilde{\xi}$ in general has two directions perpendicular to \mathbf{e}_1 we only give ξ^{\parallel} here since it is the only one that turns out to be relevant in the remainder of this work.) The derivatives of the slow-roll functions can be computed from their definitions and are given by:

$$\tilde{\epsilon}^i = 2H_a\tilde{\epsilon}(\tilde{\epsilon} + \tilde{\eta}^{\parallel}), \quad (\tilde{\eta}^{\parallel})^i = H_a[\tilde{\xi}^{\parallel} + (\tilde{\eta}^{\perp})^2 + \tilde{\epsilon}\tilde{\eta}^{\parallel} - (\tilde{\eta}^{\parallel})^2], \quad \mathcal{D}\tilde{\eta} = H_a[\tilde{\xi} + (\tilde{\epsilon} - \tilde{\eta}^{\parallel})\tilde{\eta}]. \quad (11)$$

In terms of the functions $\tilde{\epsilon}, \tilde{\eta}$ the Friedmann equation (6) and the background field equation (5) read

$$H_a = \frac{\kappa}{\sqrt{3}} b \sqrt{V} \left(1 - \frac{1}{3}\tilde{\epsilon}\right)^{-1/2}, \quad (12)$$

$$\dot{\phi}^i + \frac{2}{\sqrt{3}} \frac{1}{\kappa} b \mathbf{G}^{-1} \nabla^T \sqrt{V} = -\sqrt{\frac{2}{3}} b \sqrt{V} \frac{\sqrt{\tilde{\epsilon}}}{1 - \frac{1}{3}\tilde{\epsilon}} \left(\frac{1}{3}\tilde{\eta} + \frac{\frac{1}{3}\tilde{\epsilon}\mathbf{e}_1}{1 + \sqrt{1 - \frac{1}{3}\tilde{\epsilon}}} \right). \quad (13)$$

(Notice that for a positive potential V the function $\tilde{\epsilon} < 3$, as can be seen from its definition.) We can now define precisely what is meant by slow roll as these two background equations are still exact. Slow roll is valid if $\tilde{\epsilon}$, $\sqrt{\tilde{\epsilon}}\tilde{\eta}^{\parallel}$ and $\sqrt{\tilde{\epsilon}}\tilde{\eta}^{\perp}$ are (much) smaller than unity. For this reason $\tilde{\epsilon}$, $\tilde{\eta}^{\parallel}$ and $\tilde{\eta}^{\perp}$ are called slow-roll functions. The function $\tilde{\xi}$ is called a second order slow-roll function because it involves two slow-roll derivatives, and it is assumed to be of an order comparable to $\tilde{\epsilon}^2$, $\tilde{\epsilon}\tilde{\eta}^{\parallel}$, etc. If slow roll is valid, we can use expansions in powers of these slow-roll functions to estimate the relevance of various terms in a given expression. For example, the background field equation up to and including first order is given by (13) with the right-hand side put to zero, as all those terms are order 3/2 or higher. To first order the Friedmann equation (12) is approximated by replacing $(1 - \tilde{\epsilon}/3)^{-1/2}$ by $(1 + \tilde{\epsilon}/6)$.

At the level of the solutions of these equations we make the following definition. An approximate solution of an equation of motion is said to be accurate to first order in slow roll, if the relative difference between this solution and the exact one is of a smaller numerical order than the slow-roll functions. This relative error depends in general on the size of the integration interval. Let us explain this with the following example that will turn out to be important in section 3.3. From (11) we see that the time derivatives of the slow-roll functions are second order quantities. Hence we can make the assumption that to first order the slow-roll functions are constant. Switching to the number of e-folds N , which is related as $dN = H_a d\tau$ to the time variable τ , we can then integrate (11) to see the variation of $\tilde{\epsilon}$ over an interval $[N_1, N_2]$:

$$\Delta\tilde{\epsilon} = \int_{N_1}^{N_2} dN 2\tilde{\epsilon}(\tilde{\epsilon} + \tilde{\eta}^{\parallel}) = 2\tilde{\epsilon}_0(\tilde{\epsilon}_0 + \tilde{\eta}_0^{\parallel})(N_2 - N_1). \quad (14)$$

Here the subscript $_0$ denotes some reference time in this interval where the slow-roll functions are evaluated. Hence we see that if the interval $(N_2 - N_1)$ becomes larger than $1/(2(\tilde{\epsilon}_0 + \tilde{\eta}_0^{\parallel}))$, $\Delta\tilde{\epsilon}$ becomes larger than $\tilde{\epsilon}_0$ and the assumption of taking $\tilde{\epsilon}$ constant over this interval to first order is certainly not valid anymore. (An example of the real behaviour of the slow-roll functions can be found in figure 1b) in section 4.2.) Of course solutions of first order slow-roll equations of motion that make use of the assumption that the slow-roll functions are constant are then also no longer accurate to first order. When making use of slow-roll approximations we will pay attention to effects related to the size of the integration interval. In the literature these effects are usually ignored and the solution of an equation of motion valid to first order is (implicitly) assumed to be accurate to first order as well. However, with that assumption the numerical error between slow-roll and exact solution can become very large depending on the size of the interval of integration, which is the reason for our revised definition.

The slow-roll functions (9) are all defined as functions of covariant derivatives of the velocity ϕ^i and the Hubble parameter H_a . If the zeroth order slow-roll approximation works well, that is if the right-hand side of (13) can be neglected, as well as the $\tilde{\epsilon}$ in (12), then we can use these two equations to eliminate ϕ^i and H_a in favour of the potential V . This is the way the conventional single field slow-roll parameters are defined. However, this conventional definition has the disadvantage that the slow-roll conditions become consistency checks. While we can expand the exact equations in powers of the slow-roll functions, that is impossible by construction with the conventional slow-roll parameters.¹ In order to avoid confusion we compare the slow-roll functions we defined in (9) with the ones conventionally used in the single field case, ϵ and η :

$$\epsilon = \frac{1}{2\kappa^2} \frac{V_{,\phi}^2}{V^2} = \tilde{\epsilon}, \quad \eta = \frac{1}{\kappa^2} \frac{V_{,\phi\phi}}{V} = -\tilde{\eta}^{\parallel} + \tilde{\epsilon}, \quad (15)$$

where the last equalities in both equations are only valid to lowest order in the slow-roll approximation.

For later use we introduce the matrix Z by

$$(Z)_{mn} = -(Z^T)_{mn} = \frac{1}{H_a} \mathbf{e}_m^\dagger \mathcal{D} \mathbf{e}_n, \quad (16)$$

which shows a nice interplay between the unit vectors and the notion of slow roll. The anti-symmetry of Z follows because $(\mathbf{e}_m^\dagger \mathbf{e}_n)^i = 0$. To determine its components we observe that

$$\mathcal{D} \mathbf{e}_{n+1} \cdot \mathbf{e}_{n-p} + \mathbf{e}_{n+1} \cdot \mathcal{D} \mathbf{e}_{n-p} = 0, \quad \mathcal{D} \mathbf{e}_{n+1} \cdot \tilde{\boldsymbol{\eta}}^{(n)} + H_a \mathbf{e}_{n+1} \cdot \tilde{\boldsymbol{\eta}}^{(n+1)} = 0, \quad (17)$$

because \mathbf{e}_{n+1} is perpendicular to \mathbf{e}_{n-p} with $0 \leq p < n$ and to $\tilde{\boldsymbol{\eta}}^{(n)}$. From the construction of $\phi^{(n)}$ in (8) we see that $\mathcal{D} \mathbf{e}_n$ can never get a component in a direction higher than \mathbf{e}_{n+1} . Hence we deduce from the first equation in (17) that for $p \geq 1$, $\mathcal{D} \mathbf{e}_{n+1}$ and \mathbf{e}_{n-p} are perpendicular. Using this we see that of the first term of the second equation only the \mathbf{e}_n direction is relevant, so that the only non-zero components of Z read

$$Z_{nn+1} = -Z_{n+1n} = -\frac{\mathbf{e}_{n+1} \cdot \tilde{\boldsymbol{\eta}}^{(n+1)}}{\mathbf{e}_n \cdot \tilde{\boldsymbol{\eta}}^{(n)}}, \quad (18)$$

which is first order in slow roll.

¹In the context of single field inflation this was noted before and discussed in detail in [9].

3 Perturbations in multiple field inflation

3.1 Equations of motion for the perturbations

This section describes the coupled system of gravity, encoded by the metric $g_{\mu\nu}$, and multiple scalar field perturbations $\delta\phi$, during inflation. We separate both the scalar fields and the metric into a homogeneous background part and an inhomogeneous perturbation, which is assumed to be small. Since the observed fluctuations in the CMBR are tiny, this assumption is well-motivated. Consequently one can linearize all equations with respect to the perturbations. We define

$$\begin{aligned}\phi^{\text{full}}(\eta, \mathbf{x}) &= \phi(\eta) + \delta\phi(\eta, \mathbf{x}), \\ g_{\mu\nu}^{\text{full}}(\eta, \mathbf{x}) &= g_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \mathbf{x}) = a^2(\eta) \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} - 2a^2(\eta)\Phi(\eta, \mathbf{x}) \begin{pmatrix} 1 & 0 \\ 0 & \delta_{ij} \end{pmatrix}.\end{aligned}\tag{19}$$

As is discussed in [18], this metric is obtained by applying the so-called longitudinal gauge to the flat Robertson-Walker metric in the case when only scalar metric perturbations and a scalar matter theory are considered. In this gauge all formulae look the same as when the gauge-invariant approach [1, 18] is used. The gravitational (Newtonian) potential $\Phi(\eta, \mathbf{x})$ describes the scalar metric perturbations.

The equation of motion for the perturbations of the metric is obtained by linearizing and combining the (00) and (ii) components of the Einstein equations (4):

$$\Phi'' + 6\mathcal{H}\Phi' + 2(\mathcal{H}' + 2\mathcal{H}^2)\Phi - \Delta\Phi = -\kappa^2 a^2(\nabla V \delta\phi),\tag{20}$$

where the spatial Laplacean is given by $\Delta = \sum_i \partial_i^2$, while the integrated (0i) component of the Einstein equations leads to the constraint equation

$$\Phi' + \mathcal{H}\Phi = \frac{1}{2}\kappa^2 \phi' \cdot \delta\phi = \frac{1}{2}\kappa^2 |\phi'| |\delta\phi|.\tag{21}$$

Here we have decomposed $\delta\phi = \delta\phi^{\parallel} \mathbf{e}_1 + \delta\phi^{\perp}$.² In addition we have the equation of motion for the scalar field perturbations,

$$\left(\mathcal{D}_\eta^2 + 2\mathcal{H}\mathcal{D}_\eta - \Delta + a^2\tilde{\mathbf{M}}^2(\phi)\right) \delta\phi = 4\Phi'\phi' - 2a^2\Phi \mathbf{G}^{-1}\nabla^T V,\tag{22}$$

where we have introduced the (effective) mass-matrices

$$\tilde{\mathbf{M}}^2 \equiv \mathbf{M}^2 - \mathbf{R}(\dot{\phi}, \dot{\phi}), \quad \mathbf{M}^2 \equiv \mathbf{G}^{-1}\nabla^T \nabla V,\tag{23}$$

with \mathbf{R} the field curvature as defined in the appendix. This system of perturbation equations must be solved in the background determined by the scalar fields (5) and the Friedmann equations (6). Using the integrated (0i) component of the Einstein equations (21) together with the background equation of motion for the scalar fields (5), the right-hand side of equation (20) for Φ can be rewritten as

$$-\kappa^2 a^2(\nabla V \delta\phi) = 2(\Phi' + \mathcal{H}\Phi) \left(\frac{1}{|\phi'|} (\mathcal{D}_\eta \phi') \cdot \mathbf{e}_1 + 2\mathcal{H} \right) + \kappa^2 (\mathcal{D}_\eta \phi') \cdot \delta\phi^{\perp},\tag{24}$$

where we used the definition of the projection operators. Inserting this expression in (20) and realizing that $|\phi'|' |\phi'| = (\mathcal{D}_\eta \phi') \cdot \phi'$, we get

$$\Phi'' + 2 \left(\mathcal{H} - \frac{|\phi'|'}{|\phi'|} \right) \Phi' + 2 \left(\mathcal{H}' - \mathcal{H} \frac{|\phi'|'}{|\phi'|} \right) \Phi - \Delta\Phi = \kappa^2 (\mathcal{D}_\eta \phi') \cdot \delta\phi^{\perp}.\tag{25}$$

²A similar decomposition in the case of two field inflation was also discussed in [3].

In the single field case the right-hand side is zero because $\delta\phi^\perp$ then vanishes by construction.

The system of perturbations (25), (21) and (22) is quite complicated. To make the physical content more transparent, we introduce new variables u and \mathbf{q} (linearly related to Φ and $\delta\phi$, respectively),

$$u \equiv \frac{a}{\kappa^2 |\phi'|} \Phi, \quad \mathbf{q} \equiv a \left(\delta\phi + \frac{\Phi}{\mathcal{H}} \phi' \right), \quad (26)$$

which satisfy the following two requirements:

1. The equations of motion for both u and \mathbf{q} do not contain first order conformal time derivatives;
2. The equation of motion for \mathbf{q} is homogeneous and \mathbf{q} is gauge invariant.

The first requirement makes a direct comparison between the size of the Fourier mode $\mathbf{k}^2 = k^2$ and other physical background quantities in the equation of motion possible. In section 3.3 we make use of this to distinguish between different regions for the behaviour of the solutions. The other requirement ensures that we can naively quantize \mathbf{q} using the Lagrangean corresponding to the equation of motion for \mathbf{q} in section 3.2. As \mathbf{q} is gauge invariant and linearly related to $\delta\phi$, apart from the shift proportional to $\Phi\phi'$, no non-physical degrees of freedom are quantized. The single field version of \mathbf{q} , including its equation of motion and quantization, was first introduced by Sasaki and Mukhanov [25, 17], which is why variables of this type are sometimes referred to as Sasaki-Mukhanov variables.

To derive the equation of motion for \mathbf{q} we need an auxiliary result. By differentiating the background field equation in terms of conformal time, $\mathcal{D}_\eta \phi' + 2\mathcal{H}\phi' + a^2 \mathbf{G}^{-1} \nabla^T V = 0$, once more we obtain

$$\mathcal{D}_\eta^2 \phi' + 2(\mathcal{H}' - 2\mathcal{H}^2)\phi' + a^2 \tilde{\mathbf{M}}^2 \phi' = 0, \quad (27)$$

where we used that $\mathcal{D}_\eta(\mathbf{G}^{-1} \nabla^T V) = \mathbf{M}^2 \phi' = \tilde{\mathbf{M}}^2 \phi'$ (because of the anti-symmetry properties of the curvature tensor $\mathbf{R}(\phi, \dot{\phi})\phi' = 0$). The equation for \mathbf{q} is then obtained from the equation of motion (22) for $\delta\phi$ and (20), (21) for Φ , using the projectors (7) and slow-roll functions (9). Combining this with the derivatives of \mathcal{H} from (6) and of the slow-roll functions (11), we finally obtain the homogeneous equation for the spatial Fourier mode \mathbf{k} of \mathbf{q} :

$$\mathcal{D}_\eta^2 \mathbf{q}_\mathbf{k} + (k^2 + \mathcal{H}^2 \Omega) \mathbf{q}_\mathbf{k} = 0, \quad L = \frac{1}{2} \mathcal{D}_\eta \mathbf{q}_\mathbf{k}^\dagger \mathcal{D}_\eta \mathbf{q}_\mathbf{k} - \frac{1}{2} \mathbf{q}_\mathbf{k}^\dagger (k^2 + \mathcal{H}^2 \Omega) \mathbf{q}_\mathbf{k}. \quad (28)$$

Here L is the associated Lagrangean and

$$\Omega \equiv \frac{1}{H^2} \tilde{\mathbf{M}}^2 - (2 - \tilde{\epsilon}) \mathbb{1} - 2\tilde{\epsilon} \left((3 + \tilde{\epsilon}) \mathbf{P}^\parallel + \mathbf{e}_1 \tilde{\eta}^\dagger + \tilde{\eta} \mathbf{e}_1^\dagger \right). \quad (29)$$

The $(n1)$ components of Ω can be expressed completely in terms of slow-roll functions using

$$\frac{1}{H^2} \tilde{\mathbf{M}}^2 \mathbf{e}_1 = \frac{1}{H^2} \mathbf{M}^2 \mathbf{e}_1 = 3\tilde{\epsilon} \mathbf{e}_1 - 3\tilde{\eta} - \tilde{\xi}. \quad (30)$$

The other components can in general not be expressed in terms of the slow-roll functions introduced in the previous subsection.

For the equation of motion for u we introduce the notation

$$\theta \equiv \frac{\mathcal{H}}{a|\phi'|} = \frac{\kappa}{\sqrt{2}} \frac{1}{a\sqrt{\tilde{\epsilon}}} \Rightarrow \frac{1}{\mathcal{H}} \frac{\theta'}{\theta} = -1 - \tilde{\epsilon} - \tilde{\eta}^\parallel, \quad \frac{1}{\mathcal{H}^2} \frac{\theta''}{\theta} = 2\tilde{\epsilon} + \tilde{\eta}^\parallel + 2(\tilde{\eta}^\parallel)^2 - (\tilde{\eta}^\perp)^2 - \tilde{\xi}^\parallel, \quad (31)$$

and observe that the following relations hold for the slow-roll functions:

$$\mathcal{H}' = \mathcal{H}^2(1 - \tilde{\epsilon}), \quad \frac{|\phi'|'}{|\phi'|} = \mathcal{H}(1 + \tilde{\eta}^{\parallel}), \quad (\mathcal{D}_\eta \phi')^\perp = \frac{\sqrt{2}}{\kappa} \mathcal{H}^2 \sqrt{\tilde{\epsilon}} \tilde{\eta}^\perp. \quad (32)$$

By substituting the definitions of u and \mathbf{q} in (25), where we first rewrite the relation between Φ and u as $\Phi = \kappa \sqrt{2} \mathcal{H} \sqrt{\tilde{\epsilon}} u/a$, and using the above expressions and the derivatives of the slow-roll functions given in (11), we obtain

$$u_{\mathbf{k}}'' + \left(k^2 - \frac{\theta''}{\theta}\right) u_{\mathbf{k}} = \mathcal{H} \tilde{\eta}^\perp \mathbf{e}_2 \cdot \mathbf{q}_{\mathbf{k}}. \quad (33)$$

From this one can draw the conclusion that at the level of the equations the redefined gravitational potential u decouples from the modified perpendicular components of the field \mathbf{q}^\perp to leading order, but in first order mixing between these perturbations appears.

The equations of motion (28) and (33) show that the different spatial Fourier modes of both \mathbf{q} and u decouple. From now on we only consider one generic mode \mathbf{k} , so that we can drop the subscripts \mathbf{k} . Rewriting equation (21) in terms of the components $q_n \equiv \mathbf{e}_n \cdot \mathbf{q}$ of \mathbf{q} and differentiating it once gives

$$u' - \frac{\theta'}{\theta} u = \frac{1}{2} q_1 \quad \Rightarrow \quad u'' - \frac{\theta''}{\theta} u = \frac{1}{2} \left(q_1' + \frac{\theta'}{\theta} q_1 \right), \quad (34)$$

where θ and its derivatives are given in (31). This equation for u'' can be combined with the equation of motion (33) for u to give

$$k^2 u = \mathcal{H} \tilde{\eta}^\perp q_2 - \frac{1}{2} \left(q_1' + \frac{\theta'}{\theta} q_1 \right). \quad (35)$$

After \mathbf{q} has been quantized, this expression can be used to relate it to u .³

3.2 Quantization of the perturbations

We start with the Lagrangean (28) in terms of the basis $\{\mathbf{e}_n\}$:

$$L = \frac{1}{2} (q' + \mathcal{H} Z q)^T (q' + \mathcal{H} Z q) - \frac{1}{2} q^T (k^2 + \mathcal{H}^2 \Omega) q, \quad (36)$$

where we employ the notation $(\Omega)_{mn} = \mathbf{e}_m^\dagger \Omega \mathbf{e}_n$ and the matrix Z is given in (16). Notice that this Lagrangean has the standard canonical normalization of $\frac{1}{2} (q')^T q'$, independent of the field metric \mathbf{G} , as can be derived from the original Lagrangean (2). We maintain the vectorial structure of this multiple field system and repress the indices n, m as much as possible, which means for example that the non-bold q in this equation is a vector (in the basis $\{\mathbf{e}_n\}$). From the canonical momenta $\pi = \partial L / \partial q'^T$ we find the Hamiltonian $H = \pi^T q' - L$ and the Hamilton equations:

$$H = \frac{1}{2} (\pi - \mathcal{H} Z q)^T (\pi - \mathcal{H} Z q) + \frac{1}{2} q^T (k^2 + \mathcal{H}^2 (\Omega + Z^2)) q; \\ q' = \frac{\partial H}{\partial \pi^T} = \pi - \mathcal{H} Z q, \quad \pi' = -\frac{\partial H}{\partial q^T} = -(k^2 + \mathcal{H}^2 \Omega) q - \mathcal{H} Z \pi. \quad (37)$$

³Although this relation could in principle be used to compute u at the end of inflation, its numerical implementation can be rather awkward because of cancellation of large numbers. In numerical situations it turns out to be more convenient to determine u from its own equation of motion and only use (35) to find the correct quantization and initial conditions.

In order to avoid writing indices when considering commutation relations we use vectors α, β with components α_m, β_m in the \mathbf{e}_m basis that are independent of q and π . The canonical commutation relations can then be represented as

$$[\alpha^T \hat{q}, \beta^T \hat{q}] = [\alpha^T \hat{\pi}, \beta^T \hat{\pi}] = 0, \quad [\alpha^T \hat{q}, \beta^T \hat{\pi}] = i\alpha^T \beta. \quad (38)$$

Using the Hamilton equations it can be checked that this quantization procedure is indeed time independent. Let Q and Π be complex matrix valued solutions of the Hamilton equations, such that $q = Qa_0^* + \text{c.c.}$, $\pi = \Pi a_0^* + \text{c.c.}$ is a solution of (37) for any constant complex vector a_0 . Here c.c. denotes the complex conjugate. The Hamilton equations for Q and Π can be combined to give a second order differential equation for Q . To remove the first order time derivative from this equation, we define $\tilde{Q}(\eta) = R(\eta)\tilde{Q}(\eta)$ with R chosen such that the matrix functions R and \tilde{Q} satisfy

$$R' + \mathcal{H}ZR = 0, \quad \tilde{Q}'' + (k^2 + \mathcal{H}^2\tilde{\Omega})\tilde{Q} = 0, \quad \text{with } \tilde{\Omega} = R^{-1}\Omega R. \quad (39)$$

The matrix Π is then given by $\Pi = Q' + \mathcal{H}ZQ = R\tilde{Q}'$. We take $R(\eta_i) = \mathbb{1}$ as initial condition, since the initial condition of Q can be absorbed in that of \tilde{Q} . The equation of motion for R implies that $R^T R$ and $\ln \det R$ are constant because Z is anti-symmetric and consequently traceless. Taking into account its initial condition, it then follows that R represents a rotation.

It now follows that \hat{q} and $\hat{\pi}$ can be expanded in terms of constant creation (\hat{a}^\dagger) and annihilation (\hat{a}) operator vectors:

$$\hat{q} = Q\hat{a}^\dagger + Q^*\hat{a} = R\tilde{Q}R^{-1}R\hat{a}^\dagger + \text{c.c.}, \quad \hat{\pi} = \Pi\hat{a}^\dagger + \Pi^*\hat{a}. \quad (40)$$

The creation and annihilation operators satisfy

$$[\alpha^T \hat{a}, \beta^T \hat{a}] = [\alpha^T \hat{a}^\dagger, \beta^T \hat{a}^\dagger] = 0, \quad [\alpha^T \hat{a}, \beta^T \hat{a}^\dagger] = \alpha^T \beta. \quad (41)$$

This is consistent with the commutation relations for q and π given above, provided that the matrix functions Q and Π satisfy

$$Q^* Q^T - Q Q^{*T} = \Pi^* \Pi^T - \Pi \Pi^{*T} = 0, \quad Q^* \Pi^T - Q \Pi^{*T} = i\mathbb{1}. \quad (42)$$

These relations hold for all time, as can be checked explicitly by using the equations of motion for Q and Π to show that they are time independent, provided that they hold at some given time.

We assume that the initial state is the vacuum $|0\rangle$ defined by $\hat{a}|0\rangle = 0$ and that there is no initial particle production. This implies that the Hamiltonian initially does not contain any terms with $\hat{a}\hat{a}$ and $\hat{a}^\dagger\hat{a}^\dagger$, which leads to the condition

$$(\Pi - \mathcal{H}ZQ)^T(\Pi - \mathcal{H}ZQ) + Q^T(k^2 + \mathcal{H}^2(\Omega - Z^T Z))Q = 0. \quad (43)$$

The solution of the equations (42) and (43) can be parametrized by a unitary matrix U at the beginning of inflation, when the limit that k^2 is much bigger than any other scale is applicable:

$$Q_i = \frac{1}{\sqrt{2k}} U, \quad \Pi_i = \frac{i\sqrt{k}}{\sqrt{2}} U. \quad (44)$$

We denote expectation values with respect to the vacuum state $|0\rangle$ by $\langle \dots \rangle$. Let α, β be two vectors. Then for the expectation value of $(\alpha^T U \hat{a}^\dagger + \alpha^{*T} U^* \hat{a})^2$, with U a unitary matrix, we obtain

$$\langle (\alpha^T U \hat{a}^\dagger + \alpha^{*T} U^* \hat{a})^2 \rangle = \alpha^{*T} U^* U^T \alpha = \alpha^{*T} \alpha. \quad (45)$$

So a unitary matrix in front of the \hat{a}^\dagger will drop out in the computation of the correlator. This is even true if another state than the vacuum is used to compute the correlator. In particular this means that the correlator of the gravitational potential will not depend on the unitary matrix U in (44). To draw this conclusion we use that Q satisfies a linear homogeneous equation of motion and the relation (35) between u and \mathbf{q} . Another application of this result that will be important in section 3.4 is that the rotation matrix R in front of the \hat{a}^\dagger in (40) will drop out when computing the correlator, since a rotation matrix is also unitary.

We finish this section with some brief remarks on the assumption of taking the vacuum state to compute the correlator. The vacuum state $|0\rangle$ at the beginning of inflation seems a reasonable assumption for the calculation of the density perturbations that we can observe in the CMBR today. Even though perturbations in the CMBR have long wavelengths now, they had very short wavelengths before they went through the horizon during inflation. Therefore, their scale k at the beginning of inflation at t_i is much larger than the Planck scale. It seems a reasonable assumption that modes with momenta very much larger than the Planck scale are not excited at t_i , so that for these modes the vacuum state is a good assumption.⁴ This assumption can be tested by taking other states than the vacuum state. For instance one can try a thermal state with a temperature of the Planckian scale. Typically one finds that if there have been a few e-folds of inflation before the now observable scales went through the horizon, corrections are negligible. For a more detailed discussion on observable effects of non-vacuum initial states we refer to [7, 15].

3.3 Solutions of the perturbation equations to first order: setup

To derive an analytical expression for the gravitational correlator valid to first order in slow roll, we have to determine the evolution of the modified Newtonian potential u and quantized variables \mathbf{q} , described by the equations (33), (28) and (34), analytically and accurate up to first order during inflation. In the next section we discuss the details of the calculation, in this section we first explain the physical ideas that go into that computation. The treatment here has been partly inspired by the discussion of the transition region in [16].

Since \mathcal{H} grows rapidly, the solutions of (33) and (28) change dramatically around the time $\eta_{\mathcal{H}}$ when a scale goes through the horizon, defined by $\mathcal{H}(\eta_{\mathcal{H}}) = k$. Hence we have at least two regions of interest: the sub-horizon period ($\mathcal{H} \ll k$), when k is dominant, and the super-horizon period ($\mathcal{H} \gg k$), when k can be neglected. The question whether there is a transition region between these two, and the precise definitions of the end point of the sub-horizon region, at η_- , and the beginning of the super-horizon region, at η_+ , depend on the equations used to follow the time evolution of u and \mathbf{q} : because of the constraint (34) the system (33) and (28) is over-determined. This means that one can work with u and \mathbf{q}^\perp or with \mathbf{q} in the sub- and super-horizon regions to determine the behaviour of u and \mathbf{q} . (Using the constraint the other undetermined component can be obtained.) Although it may seem that therefore there is some arbitrariness in the final results of a calculation, this is not the case. Indeed, suppose one compares two different calculations of, say, the Newtonian potential at the end of inflation, both valid up to first order, then their results may differ only at order $3/2$.

In order to show that a calculation is accurate up to first order during the complete epoch of inflation, one has to argue that the approximations made in the different regions are precise up to first order and that they can be combined consistently. In particular, it

⁴There could be a problem with this approach, because our knowledge of physics beyond the Planck scale is extremely poor. In particular, the dispersion relation $\omega(\mathbf{k}) = k$ that we used implicitly might not be valid for large k : there might be a cut-off for large momenta. For a discussion of this trans-Planckian problem and possible cosmological consequences see [13, 14].

may happen that the approximations in the sub- and super-horizon regions that determine η_- and η_+ make the transition region where $k \sim \mathcal{H}$ too large, so that no analytic solution valid up to first order can be obtained there. We will show that with the choices we make below a consistent analytic solution valid to first order can be obtained.

We now state and motivate which equations and approximations are used in the three regions:

- **sub-horizon:** The equation (28) for \mathbf{q} is used with the term $\mathcal{H}^2\Omega$ neglected. It can be solved exactly and the quantization is straightforward.
- **transition:** We continue to use (28) but this time without neglecting any terms. However, we use that the transition region is sufficiently small so that we can take the slow-roll functions constant over this region and thereby obtain solutions for \mathbf{q} valid to first order using Hankel functions. We determine these constant slow-roll functions at the end of the transition region (η_+): near the end of the transition region the leading order solution of (28) grows proportionally to the scale factor a because of the $-2\mathbb{1}$ in (29) ($2\mathcal{H}^2 \approx a''/a$), hence there is more sensitivity to the slow-roll functions towards the end of this region.⁵
- **super-horizon:** We split the exact solution for u of equation (33) with k^2 neglected into a homogeneous part and a particular solution u_P . To work out u_P in a more explicit form slow-roll assumptions are necessary. In this region we use u since it is related to the Newtonian potential via a simple rescaling. Furthermore, if q_1 is used instead of the homogeneous part of u one has to be careful in numerical calculations because of cancellation of large numbers when Φ is calculated at the end.

Neglecting $\mathcal{H}^2\Omega$ in the sub-horizon region and k^2 in the super-horizon region is only valid up to first order in the regions before η_- and after η_+ , respectively:

$$k^2 \geq \tilde{\epsilon}^{-\frac{3}{2}}\mathcal{H}^2(\eta), \quad \eta \leq \eta_-; \quad k^2 \leq \tilde{\epsilon}^{\frac{3}{2}}\frac{\theta''}{\theta}(\eta) \approx \tilde{\epsilon}^{\frac{5}{2}}\mathcal{H}^2(\eta), \quad \eta \geq \eta_+. \quad (46)$$

Equality defines the boundaries η_{\pm} of the transition region. For notational simplicity we have used $\tilde{\epsilon}$ in these expressions instead of writing $\max(\tilde{\epsilon}, |\tilde{\eta}^{\parallel}|)$, since in the examples we considered $\tilde{\epsilon}$ is the largest slow-roll function. For the same reason we approximated $\theta''/\theta = 2\tilde{\epsilon} + \tilde{\eta}^{\parallel} + \dots$ by $\tilde{\epsilon}$ (the factor 2 in front of the $\tilde{\epsilon}$ is unimportant to first order).

The above argument of being able to neglect \mathcal{H}^2 and k^2 in the appropriate regions only forbids the transition region to be smaller than defined above, it does not forbid it to be larger. However, the larger the transition region is, the worse the approximation of taking the slow-roll functions constant in this region will be. Hence the transition region should be kept as small as possible, which fixes η_- and η_+ at the values defined in (46). Actually, we have to check if this approximation is consistent at all, i.e. if the condition below (14) for taking the slow-roll functions constant is satisfied in the transition region. With this approximation we can obtain an expression for $\mathcal{H}(\eta)$ by integrating the relation for \mathcal{H}' in (32) with respect to conformal time, while integrating once again gives the number of e-folds $N(\eta) \equiv \int_{\eta_i}^{\eta} \mathcal{H} d\eta$ to first order around $\eta = \eta_{\mathcal{H}}$:

$$\mathcal{H}(\eta) = \frac{-1}{(1 - \tilde{\epsilon}_+)\eta}, \quad N(\eta) = N_{\mathcal{H}} + \frac{1}{1 - \tilde{\epsilon}_+} \ln \frac{\mathcal{H}(\eta)}{k}. \quad (47)$$

⁵In [16] the same conclusion that the time of evaluation of the slow-roll functions should be equal to the time of matching to the super-horizon region was proved for a certain class of single field inflation models.

Here we used the freedom in the definition of conformal time to set $\eta_{\mathcal{H}} = -1/[(1-\tilde{\epsilon}_+)k]$. For the size of the transition region we then obtain $N_{+-} \equiv N_+ - N_- = -2 \ln \tilde{\epsilon}_+$. If $\tilde{\epsilon}_+ \approx 0.01$,⁶ we see from (14) that this gives a correction of order $3/2$ since $2|\ln \tilde{\epsilon}_+|$ is then numerically of the same order as $\tilde{\epsilon}_+^{-1/2}$, so that this correction can be neglected up to first order. For smaller $\tilde{\epsilon}_+$ the correction is even of higher order. The situation for the other slow-roll functions is completely analogous. Hence our assumption of taking all slow-roll functions constant over the region of horizon crossing is consistent. Notice that we have given here an argument that fixes the matching time η_+ , and the result turns out to be different from the horizon crossing time $\eta_{\mathcal{H}}$, which is the value that is usually taken, more or less arbitrarily, in the literature.

3.4 Solutions of the perturbation equations to first order: calculation

After these generalities, we now focus on the details of our calculation. Since the details of quantization were taken care of in section 3.2, we can now use the matrix function Q instead of the vector operator \mathbf{q} . In the sub-horizon region k^2 dominates by definition the terms proportional to \mathcal{H}^2 in (39), so that we easily find the solution for Q there:

$$Q(\eta) = \frac{1}{\sqrt{2k}} V(\eta) = \frac{1}{\sqrt{2k}} R(\eta) e^{ik(\eta-\eta_i)} U, \quad (48)$$

taking the initial condition (44) into account. Notice that $V(\eta)$ is unitary and hence for any $\eta < \eta_-$ the $Q(\eta)$ is of the form (44). Using the remark below (45) this shows that the sub-horizon region is irrelevant in the computation of the gravitational correlator.

In the second region, where the scale k goes through the horizon, and we have to keep both the k^2 and \mathcal{H}^2 terms in the equation for \tilde{Q} , we need an explicit expression for the Hubble parameter \mathcal{H} to construct a solution. As was argued in the previous section, we may take the slow-roll functions constant, evaluated at η_+ , and use (47) for \mathcal{H} and N . The rotation matrix R (see (39)) can here be written as $R(\eta) = \exp(N - N_-)Z_+$, using that the sub-horizon period is irrelevant, that Z contains only slow-roll functions, and that $\mathcal{H} = N'$. Up to first order the rotation of Ω is irrelevant so that $\tilde{\Omega} = \Omega_+$: all terms in (29) are of first or higher order except for $-2\mathbb{1}$, but that term is, of course, invariant under rotations. However, as we will see below, multiplication with a rather large term may have the effect that even a term of order $3/2$ can be important. Hence we will improve our approximation of $\tilde{\Omega} = \Omega_+ - (N - N_-)[\Omega_+, Z_+]$ by averaging it over the transition region:

$$\tilde{\Omega} = \frac{1}{\eta_+ - \eta_-} \int_{\eta_-}^{\eta_+} d\eta (\Omega_+ - (N - N_-)[\Omega_+, Z_+]) = \Omega_+ - [\Omega_+, Z_+], \quad (49)$$

where we used (47) and (46) and gave only the leading order correction.

Using (47) equation (39) becomes to first order of the form of a Bessel equation in the transition region:

$$\tilde{Q}'' + k^2 \tilde{Q} - \frac{\nu_+^2 - \frac{1}{4}}{\eta^2} \tilde{Q} = 0, \quad \nu_+^2 = \frac{9}{4} + 3\delta_+ - 3[\delta_+, Z_+], \quad \delta_+ = \tilde{\epsilon}_+ - \frac{\tilde{M}_+^2}{3H_+^2} + 2\tilde{\epsilon}_+ e_1 e_1^T. \quad (50)$$

⁶This estimate for $\tilde{\epsilon}_+$ is motivated by the case of a quadratic potential, where $1/\tilde{\epsilon}_+ = 2(N_{\text{end}} - N_+)$, and scales of interest have $N_{\text{end}} - N_+ \approx 55$.

The solution of this matrix equation can be written in terms of a Hankel function:⁷

$$\tilde{Q}(\eta) = \sqrt{\frac{\pi|\eta|}{4}} H_{\nu_+}^{(1)}(k|\eta|), \quad \nu_+ = \frac{3}{2} + \delta_+ - [\delta_+, Z_+]. \quad (51)$$

The matrix Q satisfies (44) at the beginning of inflation ($k|\eta| \gg 1$) because there the Hankel function can be approximated by $H_{\nu}^{(1)}(z) = \sqrt{2/(\pi z)} \exp i(z - \pi\nu/2 - \pi/4)$. This shows that the solution in the transition region is also valid in the sub-horizon period. For small $|z|$ on the other hand, $H_{\nu}^{(1)}(z) = \frac{1}{i\pi} \Gamma(\nu) \exp\{-(\ln|z/2|)\nu\}$, so that

$$Q = R \tilde{Q} R^{-1} R = \frac{1}{i\sqrt{2\pi k}} e^{-(N-N_-)Z_+} \Gamma(\nu_+) \left| \frac{k\eta}{2} \right|^{\frac{1}{2}-\nu_+} e^{(N-N_-)Z_+} e^{-(N-N_-)Z_+}. \quad (52)$$

Evaluating this at the end of the region of horizon crossing, and including only those terms that are relevant to first order, we find

$$\begin{aligned} Q_+ &= \frac{\mathcal{H}_+}{i\sqrt{2k^3}} E_+, \quad Q'_+ = \frac{\mathcal{H}_+^2}{i\sqrt{2k^3}} \left((1 - \tilde{\epsilon}_+) \mathbb{1} + \delta_+ - Z_+ + (N_{+-} - 1)[\delta_+, Z_+] \right) E_+, \\ E_+ &= \left[(1 - \tilde{\epsilon}_+) \mathbb{1} + (2 - \gamma - \ln 2 - \ln|k\eta_+|) (\delta_+ + (N_{+-} - 1)[\delta_+, Z_+]) \right] e^{-N_{+-}Z_+}, \end{aligned} \quad (53)$$

where $\gamma \approx 0.5772$ is the Euler constant and we used (47). For later convenience we have defined the matrix E_+ , which to zeroth order in slow roll is equal to the identity. The quantity $|k\eta_+|$ is given by $|k\eta_+| = \tilde{\epsilon}_+^{5/4}$ according to (46). The rotation matrix at the end of the expression for E_+ will drop out when computing the gravitational correlator, as was discussed below (45). This was the reason for writing the $R^{-1}R$ in (52). Notice that the commutator term, although in principle of second order, can contribute at first order because the prefactor $-(N_{+-} - 1) \ln|k\eta_+| \approx \frac{5}{2} \ln^2 \tilde{\epsilon}_+$ can numerically be of order $1/\tilde{\epsilon}_+$. In section 4.2 we discuss an example where this is indeed the case. This effect is caused by our result for the matching time η_+ and would be missed if $\eta_{\mathcal{H}}$ was used instead.

Next we turn to the super-horizon situation, where the total solution of (33) can be written as

$$u_{\mathbf{k}}(\eta) = u_{P\mathbf{k}}(\eta) + C_{\mathbf{k}}\theta(\eta) + D_{\mathbf{k}}\theta(\eta) \int_{\eta_+}^{\eta} \frac{d\eta'}{\theta^2(\eta')}, \quad u_{P\mathbf{k}} = \theta \int_{\eta_+}^{\eta} \frac{d\eta'}{\theta^2} \int_{\eta_+}^{\eta'} d\eta'' \mathcal{H}\theta\tilde{\eta}^{\perp} q_{2\mathbf{k}}, \quad (54)$$

with $C_{\mathbf{k}}$ and $D_{\mathbf{k}}$ integration constants. To stress that the integration constants are mode dependent we have reintroduced the \mathbf{k} subscripts here. Furthermore, $u_{P\mathbf{k}}$ denotes a particular solution of the inhomogeneous equation. The integration constant $C_{\mathbf{k}}$ is irrelevant because the function θ rapidly decays. For $D_{\mathbf{k}}$ we obtain, using (34), $D_{\mathbf{k}} = \frac{1}{2}\theta_{+q1\mathbf{k}}(\eta_+)$. With this we can give the Newtonian potential Φ as a quantum operator at late times during inflation up to first order in slow roll:

$$\hat{\Phi}_{\mathbf{k}}(t) = \frac{\kappa}{2ik^{3/2}} \frac{H_+}{\sqrt{\tilde{\epsilon}_+}} \left(A(t_+, t) e_1^T + \tilde{U}_P^T(t) \right) E_+ \hat{a}_{\mathbf{k}}^{\dagger} + \text{c.c.} \quad (55)$$

The functions $A(t)$ and $\tilde{U}_P^T(t)$ are defined as

$$A(t_+, t) = 1 - \frac{H(t)}{a(t)} \int_{t_+}^t dt' a(t'), \quad \tilde{U}_P^T q_{+ \mathbf{k}} = 2a_+ \sqrt{\tilde{\epsilon}_+} H \sqrt{\tilde{\epsilon}} u_{P\mathbf{k}}. \quad (56)$$

⁷This Bessel equation and its solution in terms of Hankel functions are well-known in the theory of inflationary density perturbations, see e.g. [10, 16] and references therein. However, in the multiple field case under consideration the order ν of the Hankel function is matrix valued. This should be considered in the usual way: defined by means of a series expansion.

In $A(t_+, t)$ we neglected one term which is exponentially suppressed with the number of e-folds. Notice that because the inhomogeneous term in the differential equation for u is proportional to $e_2^T q$, we can always take the constant operator q_+ out of the particular solution.

We now determine our end result, which is the direct inflationary contribution to the gravitational perturbations at the time of recombination when the CMBR was formed. These are the perturbations that carry directly over from inflation to the time of recombination in the gravitational potential. They include all contributions to the gravitational potential during inflation, both the so-called adiabatic and the entropy (or isocurvature) perturbations (see e.g. [3] for definitions). On the other hand, entropy perturbations after inflation, caused by the multiple fields decaying into different, decoupled particle species, which can enter into the gravitational potential after inflation or influence the CMBR in a different way [6, 8] are not considered in this paper. This allows us to look only at the homogeneous equation for Φ after inflation and neglect the effect of inhomogeneous source terms. We make use of the fact proved in [18] that also after inflation, during matter and radiation domination, the equation for the homogeneous part of the redefined gravitational potential u for the modes we are interested in is given by equation (33) with the right-hand side set to zero and k^2 neglected. Hence after inflation we can write the solution for u as (54), with different constants $\tilde{C}_{\mathbf{k}}$ and $\tilde{D}_{\mathbf{k}}$ and without the particular solution.

Matching our complete solution for Φ at the end of inflation t_e to the homogeneous solution for Φ after inflation, we find for the latter:

$$\begin{aligned}\Phi_{\mathbf{k}}(t) &= 2 \left(D_{\mathbf{k}} + \theta_e u'_{P\mathbf{k}}(\eta_e) - \theta'_e u_{P\mathbf{k}}(\eta_e) \right) A(t_e, t) \\ &= \frac{\kappa}{2ik^{3/2}} \frac{H_+}{\sqrt{\tilde{\epsilon}_+}} A(t_e, t) (e_1^T + U_{Pe}^T) E_+ a_{\mathbf{k}}^\dagger + \text{c.c.}\end{aligned}\quad (57)$$

This expression is only valid some time after t_e , since we have neglected the $\tilde{C}_{\mathbf{k}}$ term, which is suppressed by $1/a$. The function A is defined in (56), while the vector U_{Pe}^T is defined by:

$$U_{Pe}^T = 2\sqrt{\tilde{\epsilon}_+} \int_{\eta_+}^{\eta_e} d\eta \mathcal{H} \frac{\tilde{\eta}_+^\perp}{\sqrt{\tilde{\epsilon}}} \frac{a_+}{a} e_2^T Q Q_+^{-1}. \quad (58)$$

At the time of recombination we can use that $a(t) \propto t^{2/3}$ in a matter dominated universe to find $A(t_e, t_{rec}) = 3/5$. Finally the vacuum correlator of the gravitational potential including all direct inflationary contributions valid up to and including first order in slow roll at the time of recombination is:

$$\begin{aligned}\langle \Phi_{\mathbf{k}}(t_{rec})^2 \rangle &= \frac{\kappa^2}{4k^3} A(t_e, t_{rec})^2 \frac{H_+^2}{\tilde{\epsilon}_+} \left[(1 - 2\tilde{\epsilon}_+)(1 + U_{Pe}^T(U_{Pe} + 2e_1)) \right. \\ &\quad + 2\ell \left((2\tilde{\epsilon}_+ + \tilde{\eta}_+^\parallel) + U_{Pe}^T \delta_+(U_{Pe} + 2e_1) \right) \\ &\quad + 2\ell \Delta N \left(2(\tilde{\eta}_+^\perp)^2 + U_{Pe}^T [\delta_+, Z_+] (U_{Pe} + 2e_1) \right) \\ &\quad \left. + \ell^2 \left((2\tilde{\epsilon}_+ + \tilde{\eta}_+^\parallel)^2 + (\tilde{\eta}_+^\perp)^2 + U_{Pe}^T \delta_+^2 (U_{Pe} + 2e_1) \right) \right],\end{aligned}\quad (59)$$

with $\ell = 2 - \gamma - \ln 2 - \frac{5}{4} \ln \tilde{\epsilon}_+$ and $\Delta N = N_+ - N_- - 1$. The explicit multiple field terms are the contributions of the particular solution U_{Pe} , and the $(\tilde{\eta}_+^\perp)^2$ terms, which can contribute to first order because of the relative largeness of ℓ and ΔN . Notice that the dependence on the width of the transition region ΔN drops out in the single field case. Since U_{Pe} is to a large extent determined by $\tilde{\eta}_+^\perp$, as can be seen from its definition (58), we see that the behaviour of $\tilde{\eta}_+^\perp$ is crucial in determining whether multiple field effects are important.

3.5 Slow roll for the perturbations

The only quantity in (59) that is not yet expressed in terms of background quantities only is U_{Pe} , which still contains an integral over the perturbation quantity Q . By using slow roll for the perturbations, as well as for the background, we can write it in terms of background quantities only. We now justify the use of slow roll on the perturbations and make this notion more precise. Physically it represents the fact that the combination of background and perturbation modes far outside the horizon cannot be distinguished from the background. We introduce the substitutions

$$\phi \rightarrow \tilde{\phi} = \phi + \delta\phi, \quad b \rightarrow \tilde{b} = a(1 + \Phi), \quad a \rightarrow \tilde{a} = a(1 - \Phi), \quad (60)$$

where we have chosen to work with conformal time after substitution to make a direct comparison with section 3.1 possible. Notice that in this way the perturbed metric (19) is obtained. Applying these substitutions to (5) and linearizing gives the perturbation equation (22) with k^2 put to zero, including the field curvature term. At the same time, by linearizing the combination

$$\mathcal{D}H_a + 3H_a^2 - \kappa^2 b^2 V = 0 \quad (61)$$

of the Friedmann equations (6), the equation of motion (20) for Φ is obtained. In other words, for the super-horizon modes the system of background equations (13) and (61) for (ϕ, a, b) is also valid for the perturbed fields $(\tilde{\phi}, \tilde{a}, \tilde{b})$. Hence the solutions for (ϕ, a, b) and $(\tilde{\phi}, \tilde{a}, \tilde{b})$ can only differ in their initial conditions, so that the perturbation quantities $(\delta\phi, \Phi)$ are obtained by linearizing the background quantities with respect to the initial conditions:

$$\delta\phi = (\nabla_{\phi_+} \phi) \delta\phi_+, \quad \mathbf{P}^\perp \mathbf{q} = a \mathbf{P}^\perp \delta\phi. \quad (62)$$

This technique was also used in [31, 27]. Here we have put the variations of the initial conditions a_+ and b_+ equal to zero, as a simple counting argument shows that this is sufficient to generate a complete set of solutions. Now since slow roll applies to the background, it follows immediately that slow roll also governs the perturbations. This fact has been used previously in the literature, see e.g. [24, 19].

Applying slow roll to the equation of motion (39) and rewriting it in terms of the quantity $\bar{Q}_{SR} \equiv (a_+/a) Q Q_+^{-1}$, we find

$$\bar{Q}'_{SR} + \mathcal{H} \left(\frac{\tilde{\mathbf{M}}^2}{3H^2} - 2\tilde{\epsilon} e_1 e_1^T + Z \right) \bar{Q}_{SR} = 0, \quad \bar{Q}_{SR}(\eta_+) = \mathbb{1}. \quad (63)$$

Here we have used that $\mathcal{D}\tilde{Q} = \tilde{Q}' - \mathcal{H}\tilde{Q}$ and $\mathcal{D}^2\tilde{Q} = \tilde{Q}'' - 3\mathcal{H}\tilde{Q}' - (\mathcal{H}' - 2\mathcal{H}^2)\tilde{Q}$, because \tilde{Q} scales with one power of a . Integrating this equation and substituting the result into the expression (58) for U_{Pe}^T we find

$$U_{Pe}^T = 2\sqrt{\tilde{\epsilon}_+} \int_{\eta_+}^{\eta_e} d\eta \mathcal{H} \frac{\tilde{\eta}^\perp}{\sqrt{\tilde{\epsilon}}} e_2^T \exp \left[\int_{\eta_+}^{\eta} d\eta' \mathcal{H} \left(-\frac{\tilde{\mathbf{M}}^2}{3H^2} + 2\tilde{\epsilon} e_1 e_1^T - Z \right) \right] \quad (64)$$

to first order in slow roll. This expression is given in terms of background quantities only. One can easily show that to first order the $(n1)$, $n > 1$ components of the matrix inside the parentheses in the exponent are all zero, so that U_{Pe} never has a component in the e_1 direction. Because we have used slow roll in the derivation, expression (64) is in principle not valid at the very end of inflation, so that we have to take η_e at a somewhat earlier time when slow roll is still valid, if we want to use this expression. One would then miss the contribution of the particular solution u_P during the last few e-folds of inflation. However,

as we will show in our example in section 4, sometimes the contribution of u_P during the last few e-folds is negligible, as $\tilde{\eta}^\perp$ will go to zero at the end of inflation. Then we can take η_e to be the real end of inflation after all, and find the complete inflationary contribution. In the next section we show how U_{Pe} can be computed explicitly for the case of a quadratic potential on a flat field manifold using the concept of slow roll on the perturbations.

4 Illustration: scalar fields with a quadratic potential

4.1 Analytical expressions for background and perturbations

In this section we consider slow-roll inflation with scalar fields living on the flat manifold \mathbb{R}^N with a quadratic potential V . The slow-roll equation of motion and Friedmann equation for the background quantities to first order are given by

$$\dot{\phi} = -\frac{2}{\sqrt{3}\kappa} \partial^T \sqrt{V(\phi)}, \quad H = \frac{\kappa}{\sqrt{3}} \sqrt{V(\phi)} \left(1 + \frac{\tilde{\epsilon}}{6}\right), \quad V = \frac{1}{2} \kappa^{-2} \phi^T \mathbf{m}^2 \phi. \quad (65)$$

Here \mathbf{m}^2 is a general symmetric mass matrix given in units of the Planck mass κ^{-1} . The initial starting point of the field ϕ is denoted by $\phi_0 = \phi(0)$. The solution of the equation of motion (65) can be written in terms of one dimensionless positive scalar function $\psi(t)$:

$$\phi(t) = e^{-\frac{1}{2} \mathbf{m}^2 \psi(t)} \phi_0 \quad \Rightarrow \quad \dot{\psi} = \sqrt{\frac{2}{3}} \frac{2}{\kappa^2 \phi_0} \left(\hat{\phi}_0^T \mathbf{m}^2 e^{-\mathbf{m}^2 \psi} \hat{\phi}_0 \right)^{-\frac{1}{2}}, \quad (66)$$

with the initial condition $\psi(0) = 0$ and where $\hat{\phi}_0 \equiv \phi_0/\phi_0$ denotes the unit vector in the direction of the initial position in field space. In other words, we have determined the trajectory that the field ϕ follows through field space starting from point ϕ_0 . The number of e-folds $N = \int H dt$ and the slow-roll function $\tilde{\epsilon}$ can be given as a function of ψ by using (66):

$$N(\psi) = N_\infty \left(1 - \hat{\phi}_0^T e^{-\mathbf{m}^2 \psi} \hat{\phi}_0\right) - \frac{1}{12} \ln \frac{\hat{\phi}_0^T \mathbf{m}^2 e^{-\mathbf{m}^2 \psi} \hat{\phi}_0}{\hat{\phi}_0^T \mathbf{m}^2 \hat{\phi}_0}, \quad \tilde{\epsilon}(\psi) = \frac{1}{2N_\infty} \frac{\hat{\phi}_0^T \mathbf{m}^4 e^{-\mathbf{m}^2 \psi} \hat{\phi}_0}{(\hat{\phi}_0^T \mathbf{m}^2 e^{-\mathbf{m}^2 \psi} \hat{\phi}_0)^2} \quad (67)$$

with $N_\infty = \frac{1}{4} \kappa^2 \phi_0^2$. For the other slow-roll functions similar expressions can be obtained. The slow-roll limit for the total number of e-folds during inflation N_∞ is approached by taking the limit $\psi \rightarrow \infty$ in the zeroth order expression for N , i.e. the above expression without the logarithm.

It is useful to have a leading order estimate of $\tilde{\epsilon}_+$. To this end we take the zeroth order expression for $N(\psi)$ and assume that ψ is already so large at time t_+ that we can neglect all masses except the smallest one in the exponential $\exp(-\frac{1}{2} \mathbf{m}^2 \psi)$. Then we can solve for ψ_+ and insert this into the expression for $\tilde{\epsilon}(\psi)$ to find

$$e^{-m_1^2 \psi_+} = \frac{1}{\|\mathbf{E}_1\|^2} \frac{N_\infty - N_+}{N_\infty} \quad \Rightarrow \quad \tilde{\epsilon}_+ = \frac{1}{2(N_\infty - N_+)}. \quad (68)$$

Here m_1 is the smallest mass eigenvalue, \mathbf{E}_1 is the projection operator that projects on the eigenspace of m_1 and $\|\mathbf{E}_1\|^2 = \hat{\phi}_0^T \mathbf{E}_1 \hat{\phi}_0 \leq 1$. Since $N_\infty - N_+ \approx 55$ we see that $\tilde{\epsilon}_+ \sim 0.01$.

We continue by computing the particular solution U_{Pe}^T defined in (58). It turns out that in this case we can work out the integral analytically in slow roll, making use of the fact that we have obtained the slow-roll trajectories in (66). The velocity and acceleration are given by

$$\dot{\phi} = -\frac{1}{2} \dot{\psi} \mathbf{m}^2 \phi, \quad \mathcal{D}_t \dot{\phi} = -\frac{1}{2} \ddot{\psi} \mathbf{m}^2 \phi + \frac{1}{4} \dot{\psi}^2 \mathbf{m}^4 \phi, \quad (69)$$

while according to (62) we obtain $\delta\phi$ by varying ϕ with respect to the initial conditions:

$$\delta\phi = -\frac{1}{2}\delta\psi \mathbf{m}^2\phi + e^{-\frac{1}{2}\mathbf{m}^2\psi}\delta\phi_0, \quad (70)$$

where $\delta\psi$ is the function ψ varied with respect to ϕ_0 . The projector parallel to the velocity is given by $\mathbf{P}^\parallel = \mathbf{m}^2\phi\phi^T\mathbf{m}^2/(\phi^T\mathbf{m}^4\phi)$, and therefore we find that

$$\mathcal{D}_t\dot{\phi}^T\mathbf{P}^\perp\delta\phi = \frac{1}{4}\dot{\psi}^2\phi^T\left[\mathbf{m}^4 - \frac{\phi^T\mathbf{m}^6\phi}{\phi^T\mathbf{m}^4\phi}\mathbf{m}^2\right]e^{-\frac{1}{2}\mathbf{m}^2\psi}\delta\phi_0. \quad (71)$$

Here we have used that the first terms of $\mathcal{D}_t\dot{\phi}$ and $\delta\phi$ are proportional to $\dot{\phi}$ and hence are projected away, so that $\delta\psi$ drops out. We rewrite U_{Pe}^T such that we can apply this result:

$$U_{Pe}^T q_+ = 2\sqrt{\tilde{\epsilon}_+} \int_{t_+}^{t_e} dt \frac{H}{\sqrt{\tilde{\epsilon}}} \tilde{\eta}^T \mathbf{P}^\perp a_+ \delta\phi. \quad (72)$$

Substituting the definition (9) for $\tilde{\eta}$ and using (67) for $\tilde{\epsilon}$ and (69) to determine $|\dot{\phi}|$, the integral takes the form

$$U_{Pe}^T q_+ = \frac{\kappa\sqrt{\tilde{\epsilon}_+}}{\sqrt{2}} \int_{\psi_+}^{\psi_e} d\psi \frac{\phi^T\mathbf{m}^2\phi}{\phi^T\mathbf{m}^4\phi} \phi^T\mathbf{m}^4\mathbf{P}^\perp e^{-\frac{1}{2}\mathbf{m}^2(\psi-\psi_+)} a_+ \delta\phi_+. \quad (73)$$

By writing out the projector \mathbf{P}^\perp we can employ

$$\frac{1}{\phi^T\mathbf{m}^4\phi} \left[\phi^T\mathbf{m}^4 - \frac{\phi^T\mathbf{m}^6\phi}{\phi^T\mathbf{m}^4\phi} \phi^T\mathbf{m}^2 \right] e^{-\frac{1}{2}\mathbf{m}^2\psi} = -\frac{\partial}{\partial\psi} \left[\frac{\phi^T\mathbf{m}^2 e^{-\frac{1}{2}\mathbf{m}^2\psi}}{\phi^T\mathbf{m}^4\phi} \right] \quad (74)$$

to perform a partial integration to express U_{Pe}^T as

$$U_{Pe}^T q_+ = \frac{\kappa\sqrt{\tilde{\epsilon}_+}}{\sqrt{2}} \left[\phi^T\mathbf{P}^\perp e^{-\frac{1}{2}\mathbf{m}^2(\psi-\psi_+)} \right]_{\psi_+}^{\psi_e} a_+ \delta\phi_+. \quad (75)$$

To determine $a_+\delta\phi_+$ we use the definition of q in (26): $q_+ = a_+(\delta\phi_+ + (\sqrt{2\tilde{\epsilon}_+}/\kappa)\Phi e_1)$, where we also inserted the definition of $\tilde{\epsilon}$. Using (26) and (35) to relate Φ_+ to q_+ we obtain

$$a_+\delta\phi_+ = X^T q_+, \quad X = 1 - \tilde{\epsilon}_+^{-3/2} \left(2\tilde{\eta}_+^\perp e_2 + \left(2\tilde{\epsilon}_+ + \tilde{\eta}_+^\parallel - \delta_+ + Z_+ - \Delta N[\delta_+, Z_+] \right) e_1 \right) e_1^T, \quad (76)$$

where we made use of (53) and the definition of t_+ , $\mathcal{H}_+^2/k^2 = \tilde{\epsilon}_+^{-5/2}$, as well. With this we find our final result for U_{Pe}^T :

$$U_{Pe}^T = \frac{\kappa\sqrt{\tilde{\epsilon}_+}}{\sqrt{2}} \left(-\phi_+^\perp + X e^{-\frac{1}{2}\mathbf{m}^2(\psi_e-\psi_+)} \phi_e^\perp \right)^T. \quad (77)$$

Here all terms are written in terms of the basis $\{\mathbf{e}_n\}$: ϕ^\perp denotes the vector with components $\mathbf{e}_n^\dagger \mathbf{P}^\perp \phi$ and $\exp(-\mathbf{m}^2\psi)$ the matrix with components $\mathbf{e}_m^\dagger \exp(-\mathbf{m}^2\psi) \mathbf{e}_n$. The second term within the parentheses in the expression for U_{Pe}^T is in general very small. In the first place all but the least massive field will in general have reached zero near the end of inflation, so that ϕ_e^\perp is small. In the second place this term is suppressed by the large negative exponential, since ψ_e is very large near the end of inflation, even though we may not be able to take the limit of $\psi_e \rightarrow \infty$ since slow roll is then not valid anymore.

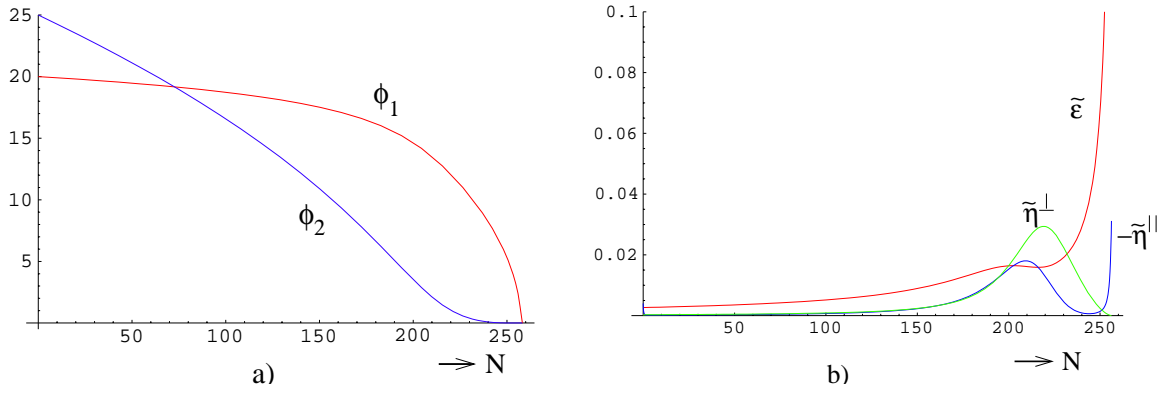


Figure 1: a) Background fields and b) slow-roll functions as a function of the number of e-folds in the model with two fields on a flat manifold with a quadratic potential with masses $m_1 = 1 \cdot 10^{-5}$, $m_2 = 2.5 \cdot 10^{-5}$ and initial conditions $\phi_1 = 20$, $\phi_2 = 25$.

4.2 Numerical example

We now treat a numerical example, not only to illustrate the theory, but also to check our analytical results. We take the situation of two fields, with masses $m_1 = 1 \cdot 10^{-5}$ and $m_2 = 2.5 \cdot 10^{-5}$ in units of the Planck mass. As initial conditions we choose $\phi_1 = 20$ and $\phi_2 = 25$, also in Planckian units. Then $N_\infty = 256.25$, while an exact numerical calculation gives a total amount of inflation of 257.8 e-folds before the oscillations start. We have chosen the overall normalization of the masses such that we get the correct order of magnitude for the amplitude of the density perturbations. Apart from giving sufficient inflation, the specific choice of initial conditions has no special meaning. We compute all background quantities exactly, as we want to check the accuracy of our analytical results for the perturbations. In figure 1 we have plotted the fields and slow-roll functions as a function of the number of e-folds. We see that the more massive field goes to zero more quickly than the less massive field, as expected from (66). Moreover, around the time that the second field reaches zero, all slow-roll functions show a bump. For the chosen masses and initial conditions the bumps are located during the last 60 e-folds. As mentioned below (59), for multiple field effects to be important, we need $\tilde{\eta}^\perp$ to be substantial during the last 60 e-folds. Hence this is a good model to look for multiple field effects. Moreover, as we see from the figure, $\tilde{\eta}^\perp$ goes to zero at the end of inflation, so that we expect corrections to U_{Pe} caused by the break-down of slow roll at the end of inflation to be small. Indeed, figure 2 shows that the contribution to U_{Pe} during the last few e-folds of inflation is negligible.

The results for the perturbations are summarized in table 1. We split the contributions to the total correlator of the gravitational potential into a homogeneous part (all terms without U_{Pe}) and a particular part (the rest, so including mixing terms). The homogeneous part is further separated into an effectively multiple field part (the $(\tilde{\eta}_+^\perp)^2$ terms) and an effectively single field part. Everything is evaluated for the mode k that crosses the horizon 60 e-folds before the end of inflation. The last column gives the relative error between our first order analytical results (59) and (77) on the one hand, and the exact numerical result on the other. For the homogeneous part the relative error is indeed of the order of $\tilde{\epsilon}_+^{3/2}$, as was our claim. Since the accuracy of the slow-roll approximations made nearer the end of inflation to compute the particular solution U_{Pe} cannot be guaranteed to this level, it is not surprising that the relative error in the total result is somewhat larger. The column before that shows the relative contributions of the various parts to the total correlator. We see that the particular solution terms are responsible for almost half the total result in this model. Hence neglecting these terms to leading order, which might naively be done because

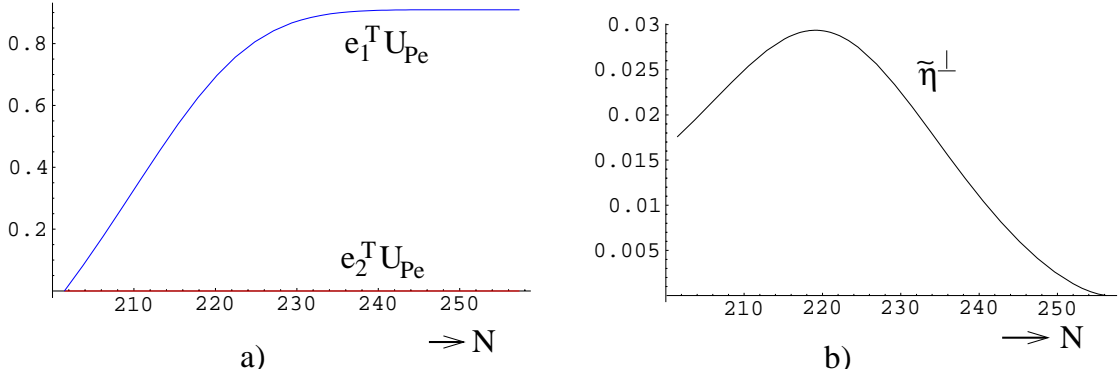


Figure 2: a) The particular contribution U_{Pe} to the gravitational correlator during the super-horizon region as a function of the number of e-folds. To show the relation with the behaviour of $\tilde{\eta}^\perp$, this slow-roll function has been plotted again in figure b), on the same horizontal scale as figure a).

	Amplitude $ \delta_{\mathbf{k}} ^2$	Contribution to total	Relative error
Homogeneous	$1.60 \cdot 10^{-9}$	0.536	0.003
single field	$1.52 \cdot 10^{-9}$	0.508	-
multiple field	$0.08 \cdot 10^{-9}$	0.028	-
Particular	$1.39 \cdot 10^{-9}$	0.464	0.061
Total	$2.99 \cdot 10^{-9}$	1	0.028

Table 1: The amplitude of the gravitational correlator $|\delta_{\mathbf{k}}|^2 = \frac{1}{2\pi^2} k^3 \langle \Phi_{\mathbf{k}}(t_{rec})^2 \rangle$ is separated into a pure homogeneous and a (mixed) particular part. The former can be divided into effective single and true multiple field contributions. The first two columns give their values and their relative contributions to the total correlator according to our analytical slow-roll result (59) combined with (77). The final column shows the relative error between these expressions and the exact numerical results. (For the latter a separation between single and multiple field homogeneous parts is not meaningful.)

they couple with a $\tilde{\eta}^\perp$ in (33), can be dangerous. The contribution of the other explicitly multiple field terms, those with $(\tilde{\eta}_\pm^\perp)^2$, is seen to be indeed of the order of $\tilde{\epsilon}_+$, and hence should be taken into account when working to first order in slow roll.

For (significantly) larger or smaller mass ratios, $\tilde{\eta}^\perp$ is smaller during the last 60 e-folds and the contribution of the explicit multiple field terms to the total correlator is less important. This could also be expected a priori, since a much larger mass ratio means that the heavy field has already reached zero before the last 60 e-folds, and the situation is effectively single field. On the other hand, a much smaller mass ratio means that we approach the limit of equal masses, which corresponds with a central potential that is also effectively single field.

5 Conclusions and discussion

We have analyzed scalar perturbations on a flat Robertson-Walker spacetime in the presence of multiple scalar fields that take values on a (curved) field manifold during slow-roll inflation. These scalar perturbations are calculated to first order in slow roll. In particular we compute the vacuum correlator of the gravitational potential in terms of background

quantities only, which is related to the temperature fluctuations that are observed in the CMBR.

A discussion of the background scalar fields served as the foundation for this analysis. The first of three central ingredients for this discussion is the manifestly covariant treatment with respect to reparameterizations of the field manifold and of the time variable. Secondly, the field dynamics (the field velocity, acceleration, etc.) naturally induce an orthonormal basis ($\mathbf{e}_1, \mathbf{e}_2, \dots$) on the field manifold. This makes a separation between effective single field and true multiple field contributions possible. Finally, we modified the definitions of the well-known slow-roll parameters to define slow-roll functions in terms of derivatives of the Hubble parameter and background field velocity for the case of multiple scalar field inflation. These slow-roll functions are vectors, which can be decomposed in the basis induced by the field dynamics. For example, the slow-roll function $\tilde{\eta}^\perp$ measures the size of the acceleration perpendicular to the field velocity. Because we did not make the assumption that slow roll is valid in the definition of the slow-roll functions, it is often possible to identify these slow-roll functions in equations of motion and make decisions about neglecting some of the terms. However, more important for precision calculations are estimates of the accuracy of the solutions of these approximated slow-roll equations; it turns out that if the size of the region of integration is too large this accuracy may be compromised.

Our calculation of the scalar perturbations accurate to first order in slow roll is based on the following cornerstones. We generalized the combined system of gravitational and matter perturbations of Mukhanov et al. [18] by defining the Mukhanov-Sasaki variables as a vector on the scalar field manifold. The decomposition of these variables in the basis induced by the background field dynamics is field space reparameterization invariant, and the corresponding Lagrangean takes the standard canonical form, making quantization straightforward. The gravitational potential only couples to the scalar field perturbation in the direction \mathbf{e}_2 with a slow-roll factor $\tilde{\eta}^\perp$.

To obtain analytic solutions for the scalar perturbations to first order in slow-roll, it is crucial to divide the inflationary epoch into three different regimes, which reflects the change of behaviour for a given mode: sub-horizon, horizon crossing (transition), and super-horizon. The matching times η_- and η_+ between these regions determine the size of the transition region. It is bounded from below by the requirement that in the sub- and super-horizon region simplifying approximations can be made valid to first order in slow roll. On the other hand, in the transition region it is essential that the slow-roll functions can be treated as constants so that the solutions can be given in terms of Hankel functions, which leads to an upper bound on the size of the transition region. The consistency of these conditions fixes the matching times uniquely to first order. We also proved that the sub-horizon region is irrelevant for the correlator of the gravitational potential.

For the homogeneous part of the gravitational potential this treatment is sufficient, but for the particular part (and for the other scalar perturbations, in general) we need a final cornerstone: the application of slow roll to the perturbations. For this it was essential that we treated the background using an arbitrary time variable. With this method an integral expression of the particular solution in terms of background quantities only was obtained. Although this expression is accurate to first order in slow roll as defined at the level of the equations, it is not possible to reach the same level of (numerical) accuracy as for the homogeneous solution, since we have to use slow roll during the later stages of inflation when the slow-roll functions become larger. However, if $\tilde{\eta}^\perp$ goes to zero at the end of inflation the accuracy can still be sufficient.

Finally, we discussed the example of multiple scalar fields on a flat manifold with a quadratic potential. To first order the trajectory of all fields through field space can be found in terms of one function of time, and the particular solution can be determined completely

analytically using the slow-roll approximation on the perturbations. We concluded with an explicit numerical check and found this to be consistent with our analytical results.

Multiple field effects are important in the correlator of the gravitational potential if $\tilde{\eta}^\perp$ is sizable during the last 60 e-folds of inflation. The obvious source of multiple field effects is the particular solution of the gravitational potential. We found in our numerical example that this term can contribute even at leading order. Hence it can be dangerous to neglect this term, even when looking only at leading order, as is done e.g. in [6]. This contribution is included implicitly in the function $N(\phi)$ of [21]. The rotation of the basis induced by the background field dynamics over the transition region gives rise to an additional important contribution. This subtle effect was not taken into account in [21] because they chose as basis the eigenvectors of the matrix ϵ_{ab} (corresponding with our matrix δ) evaluated at a fixed time, and neglected the effect that these are not constant in time. Although a priori a second order term, it is crucial to realize that this term, as well as other explicit multiple field terms, are multiplied by a product of two logarithms of slow-roll functions, see (59), so that they contribute at first order after all. For these logarithms it is essential that the matching time is η_+ and not the time of horizon crossing ($\eta_{\mathcal{H}}$) as is often assumed. (In the case of a single scalar field, however, we have checked numerically that the difference caused by the use of $\eta_{\mathcal{H}}$ instead of η_+ turns out to be smaller than first order.)

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A Geometrical concepts

Consider a real manifold \mathcal{M} with metric \mathbf{G} and local coordinates $\phi = (\phi^a)$. From the components of this metric G_{ab} the metric-connection Γ_{bc}^a is obtained using the metric postulate. The curvature tensor of the manifold can be introduced using tangent vectors $\mathbf{B}, \mathbf{C}, \mathbf{D}$:

$$[\mathbf{R}(\mathbf{B}, \mathbf{C})\mathbf{D}]^a \equiv R_{bcd}^a B^b C^c D^d \equiv (\Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{bd}^e \Gamma_{ce}^a - \Gamma_{bc}^e \Gamma_{de}^a) B^b C^c D^d. \quad (78)$$

One should realize that for notational convenience we do not use the standard definition as made for example in [20]: our $\mathbf{R}(\mathbf{B}, \mathbf{C})\mathbf{D}$ is conventionally denoted by $\mathbf{R}(\mathbf{C}, \mathbf{D}, \mathbf{B})$.

The metric \mathbf{G} introduces an inner product and the corresponding norm on the tangent bundle of the manifold:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^\dagger \mathbf{B} \equiv \mathbf{A}^T \mathbf{G} \mathbf{B} = A^a G_{ab} B^b, \quad |\mathbf{A}| \equiv \sqrt{\mathbf{A} \cdot \mathbf{A}}, \quad (79)$$

for any two vector fields \mathbf{A} and \mathbf{B} . The cotangent vector \mathbf{A}^\dagger is defined by $(\mathbf{A}^\dagger)_a \equiv A^b G_{ba}$. The Hermitean conjugate \mathbf{L}^\dagger of a linear operator $\mathbf{L} : T_p \mathcal{M} \longrightarrow T_p \mathcal{M}$ with respect to this inner product is defined by

$$\mathbf{B} \cdot (\mathbf{L}^\dagger \mathbf{A}) \equiv (\mathbf{L} \mathbf{B}) \cdot \mathbf{A}, \quad (80)$$

so that $\mathbf{L}^\dagger = \mathbf{G}^{-1} \mathbf{L}^T \mathbf{G}$. A Hermitean operator \mathbf{H} satisfies $\mathbf{H}^\dagger = \mathbf{H}$. An important example of Hermitean operators are the projection operators. Apart from being Hermitean, a projection operator \mathbf{P} is idempotent: $\mathbf{P}^2 = \mathbf{P}$.

To complete our discussion on the geometry of \mathcal{M} we introduce different types of derivatives. In the first place we have the covariant derivative on the manifold, denoted by ∇_a , which acts in the usual way, i.e.

$$\nabla_b A^a \equiv A_{,b}^a + \Gamma_{bc}^a A^c \quad (81)$$

on a vector A^a . On a scalar function V , the derivative ∂ and the covariant derivative ∇ are equal $(\nabla V)_a = (\partial V)_a \equiv V_{,a}$. If we represent $d\phi$ as a standing vector, ∇ and ∂ are naturally lying vectors and therefore ∇^T and ∂^T are standing vectors. The second covariant derivative of a scalar function V is a matrix with two lower indices: $(\nabla^T \nabla V)_{ab} = \nabla_a \nabla_b V$. The covariant derivative \mathcal{D}_μ with respect to the spacetime variable x^μ on a vector \mathbf{A} of the tangent bundle is defined in components as

$$\mathcal{D}_\mu A^a \equiv \partial_\mu A^a + \Gamma_{bc}^a \partial_\mu \phi^b A^c, \quad (82)$$

while \mathcal{D}_μ acting on a scalar is simply equal to ∂_μ . Notice that the spacetime derivative of the background field $\partial_\mu \phi$ and the field perturbation $\delta\phi$ transform as vectors, even though the fields ϕ in general do not, as they are coordinates on a manifold.

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